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Abstract
In this work we discuss some recent contributions to the random search problem.
Our analysis includes superdiffusive Lévy processes and correlated random
walks in several regimes of target site density, mobility and revisitability.
We present results in the context of mean-field-like and closed-form average
calculations, as well as numerical simulations. We then consider random
searches performed in regular lattices and lattices with defects, and we discuss
a necessary criterion for distinguishing true superdiffusion from correlated
random walk processes. We invoke energy considerations in relation to critical
survival states on the edge of extinction, and we analyze the emergence of Lévy
behavior in deterministic search walks. Finally, we comment on the random
search problem in the context of biological foraging.

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1. Introduction

What is the most efficient strategy to adopt while searching for randomly placed sites whose
locations are not *a priori* known for situations when the searcher has information only about
its close vicinity? This *random search problem* [1–8] has attracted continuous interest over
the past three decades, in part due to its broad interdisciplinary range of applications, which
include animal foraging [1–8], automated computer searches of registers in high-capacity
databases [9], oil recovery from mature reservoirs [10] and facilitated target location in
biological systems [11] (e.g. enhanced diffusion of regulatory proteins while 'searching' for
specific DNA target sites). Recently, this problem has also found interesting connections

with human motion and related topics [12]. Concepts such as search strategy and efficiency make the approach to the problem naturally fit with the concepts and methods of statistical physics. Indeed, the input of concepts such as random walks, fractal geometry, scaling ideas, correlation functions, first-passage time processes and superdiffusive dynamics has rendered great progress to this field. Actually, Lévy flights and walks [1–6], intermittent walks [13–16] and correlated random walks (CRW) [17, 18] are among the most studied random walk models proposed as the best strategy in the random search problem.

Here we discuss some recent contributions which attempt to understand the mechanisms underlying efficient searches. Most of the results concern Lévy flights and walks, which are characterized by the existence of rare but extremely large steps, alternating between sequences of many short-length jumps [1]. This property leads to these processes presenting anomalous (superdiffusive) dynamics, in contrast to the diffusive dynamics observed in Brownian walks, which are governed by the central limit theorem. The generalized central limit theorem [19] implies that after a large number N of steps, the probability density of a displacement ℓ in a Lévy flight process converges to a Lévy stable distribution:

$$p(\ell, N) = \frac{1}{\pi} \int_0^\infty \exp\left(-N\ell_0^\alpha q^\alpha\right) \cos(q\ell) \, \mathrm{d}q, \tag{1}$$

where ℓ_0 is the characteristic width of the distribution of a single step, and $0 < \alpha \leqslant 2$. Equation (1) is a generalization of the Gaussian distribution, and for $\alpha < 2$ leads to a power-law decay with distance for large ℓ:

$$p(\ell, N) = \frac{N\Gamma(\mu) \sin(\pi(\mu - 1)/2)}{\pi \ell_0} \left(\frac{\ell_0}{\ell}\right)^\mu + O(\ell^{-2\mu+1}), \tag{2}$$

where $\mu = \alpha + 1$, and $\Gamma(x)$ represents the gamma function. Such a broad distribution of step length ℓ implies diverging mth order moments, for $m \geqslant \mu - 1$ and $\mu < 3$. As a consequence, the mean-square displacement for a Lévy flight does not exist as a function of time if one considers the definition in which the duration of each step is constant, so that the velocity and time of travel are, respectively, proportional to the step length and number of steps.

In contrast, in Lévy walks the constant velocity implies the time of travel proportional to the total path length, causing the mean-square displacement to grow with time superlinearly— i.e., in a 'superdiffusive' fashion. The total path length of Lévy flights then corresponds to the total time of travel in Lévy walks. Also, the mean first passage time for Lévy flights and walks are, respectively, equivalent to the average number and total length of flights taken by the searcher before finding the next target site [20, 21].

In most of this work, we consider a random walker with statistically independent step lengths chosen from the probability density distribution

$$p(\ell) = \frac{(\mu - 1)\ell_0^{\mu-1}}{2} \frac{1}{\ell^\mu}, \qquad \ell \geqslant \ell_0, \tag{3}$$

and $p(\ell) = 0$ otherwise, with the lower cutoff ℓ_0 representing the minimum flight length. By comparing equations (2) and (3), we note [20] that the choice, $N^{-1} = 2\Gamma(\alpha + 1) \sin(\pi\alpha/2)/(\pi\alpha)$, guarantees that the asymptotical behavior of equation (1) is identical to that of equation (3). Moreover, as the second moment of distribution (3) diverges for $1 < \mu \leqslant 3$, the central limit theorem does not hold, and anomalous (superduffisive) dynamics takes place. Indeed, Lévy walks and flights are related to a Hurst exponent [1] $H > 1/2$, whereas Brownian behavior (diffusive walks with $H = 1/2$) emerges for $\mu > 3$ [22]. In particular, for Lévy walks one finds $H = 1$ for $1 < \mu \leqslant 2$, with ballistic dynamics emerging as $\mu \to 1$ (the case $\mu = 2$ corresponds to the Cauchy distribution). For $\mu \leqslant 1$ distribution (3) is not normalizable. The power-law dependence on ℓ implies the self-affine
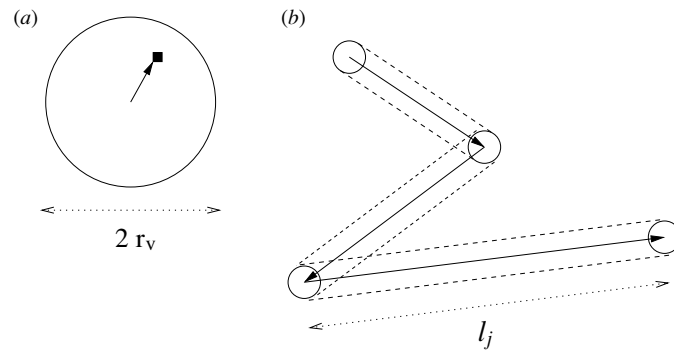
Figure 1. Random search model [23]: (*a*) if a target site (solid square) is located within a 'direct vision' distance r_v, then the searcher moves on a straight line to it; (*b*) otherwise, the searcher chooses a random direction and a step length ℓ_j from equation (3) and incrementally moves, continually looking for a target within a radius r_v along its way.

property $p(\gamma \ell) = \gamma^{-\mu} p(\ell)$, valid down to the scale of the lower cutoff, thus leading to step lengths without a characteristic scale and scale-invariant random search walks with fractal dimension, $d_f = \mu - 1$, $1 < \mu \leqslant 3$, of the set of visited sites [1].

The structure of the paper is as follows. In section 2, we present a random search model, based on the step length distribution (3), and discuss its implications in several regimes of target site density, mobility, size and revisitability, in the context of a mean-field-like analytical solution and results from numerical simulations. Section 3 is devoted to the analytical calculation of relevant quantities, such as the mean number and the average total length (related to the inverse of the search efficiency) of a Lévy flight between two consecutive target sites in one dimension. In section 4, we discuss a random search in regular and defective lattices, rather than in continuous space. We discuss in section 5 a necessary criterion for distinguishing true superdiffusion from correlated random walk processes. We discuss energy considerations in section 6 and critical survival states on the edge of extinction. We analyze the possible emergence of Lévy behavior in deterministic walks in section 7. We make contact with biological foraging in section 8.

2. A Lévy-flight-based model for random searches

We begin by considering the random search model [23] with the following general strategy rules for a searcher to look for randomly placed target sites in a *D*-dimensional search space (see figure 1):

(1) If there is a target site located within a 'direct vision' distance r_v, then the searcher moves on a straight line to the nearest target site.

(2) If there is no target site within a distance r_v, then in its *j*th step the searcher chooses a direction at random and a distance ℓ_j from the probability distribution $p(\ell_j)$, equation (3), valid for all *j*. It then incrementally moves to the new point, continually looking for a target within a radius r_v along its way. If it does not detect a target, it stops after traversing the distance ℓ_j and chooses a new direction and a new distance ℓ_{j+1}. Otherwise it proceeds to the target as in rule (1).

As rule (2) indicates, the search path might contain a number of steps truncated by the finding of a target. As a consequence, the power-law decay with distance implied by

equation (3) cannot extend all the way to infinity, and an effective *truncated* Lévy distribution [24] arises from this set of rules. Indeed, as we discuss below, truncated step length distributions may play an important role in the random search problem.

The efficiency of the search, η, is defined [23] as the ratio of the number of target sites visited to the total distance traversed by the forager. Since this distance is equal to the product of the number of targets visited and the average distance traversed between two successive target sites, $\langle L \rangle$, then

$$\eta = \frac{1}{\langle L \rangle}. \tag{4}$$

Although quite general and idealized, the above model allows us to explore the essence of the statistical physics concepts related to random searches discussed in section 1, preserving at the same time the main dynamical aspects of searches. The above rules, nevertheless, are not sufficient to fully formulate the problem. Since a great deal of motivation to the random search problem arises in the context of biological foraging, a number of ideas associated with more realistic search situations in nature have been incorporated into the above rules, e.g., target revisitability [23]. In the case of a 'non-destructive search', the searcher can visit the same target site any number of times. In contrast, in a 'destructive search' a target site once found by the searcher becomes undetectable in subsequent flights; in this case, a new target site is randomly placed elsewhere in order to keep the density of targets unaltered. In the intermediate regime of a regenerative search [25, 26], targets reappear and become available again after some finite characteristic time. These regenerative searches might represent the most realistic type of search, such as in the context of biological foraging. Nonetheless, some ingredients are still not taken into account in these rules, such as predator–prey relationships and learning processes. However, from the results reviewed below, it has been argued [27] that the model seems to be capable of capturing many of the main features of biological foraging.

We consider a 'mean-field-like' approach to the above model [23]. By approximating the average distance traversed between two successive target sites as the product of the average length of a single flight, $\langle \ell \rangle$, and the mean number of flights taken by the searcher between two successive target sites, N, we find

$$\eta \approx \frac{1}{N \langle \ell \rangle}. \tag{5}$$

Let λ be the 'mean free path' associated with the random distribution of target sites. In a two-dimensional landscape,

$$\lambda = (2 r_v \rho)^{-1}, \tag{6}$$

where ρ is the target site area density. On the other hand, the approximate average length of a single flight reads

$$\langle \ell \rangle \approx \frac{\int_{r_v}^{\lambda} \mathrm{d}x \, x^{1-\mu} + \lambda \int_{\lambda}^{\infty} x^{-\mu} \, \mathrm{d}x}{\int_{r_v}^{\infty} x^{-\mu} \, \mathrm{d}x} = \left(\frac{\mu - 1}{2 - \mu} \right) \left(\frac{\lambda^{2-\mu} - r_v^{2-\mu}}{r_v^{1-\mu}} \right) + \frac{\lambda^{2-\mu}}{r_v^{1-\mu}}, \tag{7}$$

where in the second term it is assumed [23], as in a mean-field-like distribution of targets, that a new target site is always encountered (truncated step) a maximum distance λ away from the previously visited target.

Consider first the case of a destructive search. The mean number of flights, N_d, taken to cover an average distance λ between two successive target sites scales as [23, 28]

$$N_d \sim (\lambda / r_v)^{\mu - 1}, \tag{8}$$

for $1 < \mu \leqslant 3$, whereas

$$N_d \sim (\lambda/r_v)^2 \tag{9}$$

for $\mu > 3$ (normal diffusion). The scaling arguments leading to this expression are as follows [6]. For a Lévy flight the pseudomean-squared displacement scales with time t as

$$\langle |x|^\delta \rangle^{2/\delta} \sim t^{2/\alpha}, \tag{10}$$

where $\alpha = \mu - 1$ denotes the Lévy index and $\delta < \alpha$. By identifying N_d with t and λ with the (pseudo)root-mean-square displacement corresponding to a target, one obtains the above result. Note that $\mu - 1$, for $1 < \mu \leqslant 3$, is the fractal dimension of the set of sites visited by a Lévy random walker [1].

On the other hand, in the case of a non-destructive search, since previously visited sites can then be revisited, the mean number N_d of flights between successive target sites in equation (8) overestimates the number N_n for the non-destructive case. We assume that $N_n \sim N_d^{1/2}$ holds generally, so it follows that [23]

$$N_n \sim (\lambda/r_v)^{(\mu-1)/2}, \tag{11}$$

for $1 < \mu \leqslant 3$, whereas

$$N_n \sim \lambda/r_v \tag{12}$$

for $\mu > 3$ (normal diffusion). It has been shown that equation (11) is rigorous [20] (see also section 3).

In the case in which target sites are plentiful, i.e., $\lambda \lesssim r_v$, then

$$\langle \ell \rangle \sim \lambda \tag{13}$$

and $N_d \sim N_n \sim 1$, so that η becomes essentially independent of μ. This means that there is no evident advantage in taking long flights when targets are plentiful, since the finding of target sites is mostly driven by detection in the close vicinity of the searcher (rule (1)). As a consequence, the difference between the efficiency of (truncated) Lévy ($1 < \mu \leqslant 3$) and Gaussian ($\mu > 3$) search strategies becomes negligible.

Now we consider the more interesting case in which the target sites are sparsely distributed, i.e. $\lambda \gg r_v$. In the destructive search, by substituting equations (7) and (8) into equation (5) we find that the mean efficiency η is larger for lower values of μ. Actually, when

$$\mu = 1 + \epsilon, \qquad \epsilon \to 0^+, \tag{14}$$

the fraction of flights with $\ell_j < \lambda$ becomes negligible, and the forager effectively moves along straight lines (ballistic motion) until it detects a target site. In contrast, in the non-destructive case with low density of targets, substitution of equations (7) and (11) into equation (5) leads to the optimal efficiency achieved at [23]:

$$\mu_{\text{opt}} = 2 - \delta, \tag{15}$$

where

$$\delta \sim 1/[\ln(\lambda/r_v)]^2, \tag{16}$$

so that $\delta \to 0$ as λ/r_v increases (see figure 2). Thus, in the absence of *a priori* knowledge about the locations of the randomly distributed target sites, an optimal strategy for a searcher is to choose $\mu_{\text{opt}} \approx 2$ when λ/r_v is large and sites can be revisited any number of times. In this case, the optimal search strategy arises as a compromise between the trends to return to the last visited site (favored by strategies with $\mu \to 3$) and to leave to look for new sites (favored by strategies with $\mu \to 1$).

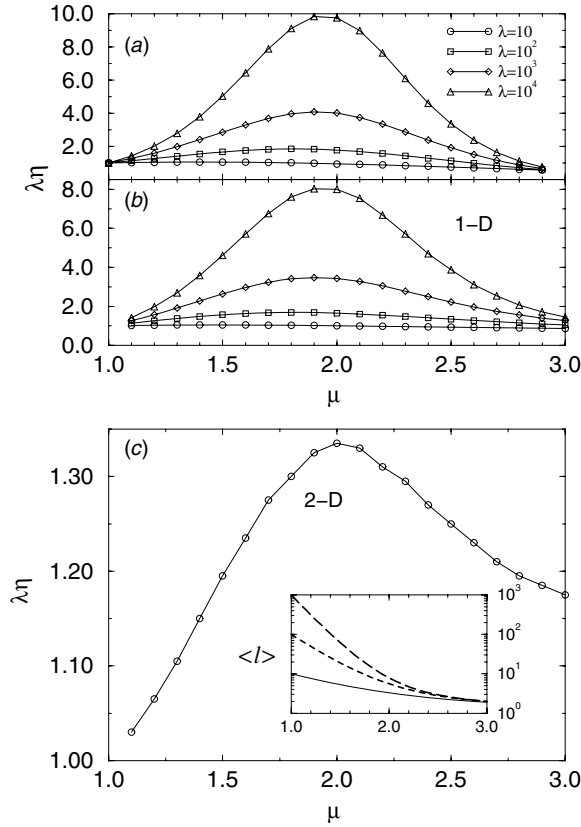**Figure 2.** Product of the mean free path λ and the search efficiency η plotted against the Lévy index μ in one dimension [23], for the case of a non-destructive search: (*a*) from equations (5) to (8) and (*b*) from numerical simulations. (*c*) $\lambda\eta$ in two dimensions, for $\lambda = 5000$; the inset shows $\langle \ell \rangle$ as a function of μ for $\lambda = 10$ (solid line), $\lambda = 10^2$ (dashed line) and $\lambda = 10^3$ (long-dashed line). All data are obtained using $r_v = 1$.

We next consider the general intermediate regenerative search regime, in which a previously visited site becomes available for future visits after some typical target-regeneration delay time τ [25, 26]. We first note that the non-destructive and destructive cases mentioned above correspond to $\tau \to 0$ and $\tau \to \infty$, respectively. Also, by noting that the mean number of flights between two successive target sites increases monotonically with τ (for fixed μ and λ/r_v), we can define an arbitrary function $\Gamma(\tau)$, monotonically decreasing with τ, so that the mean number of flights between two successive target sites can be written as

$$N_r \sim \lambda^{(\mu-1)/\Gamma}. \tag{17}$$

The destructive and non-destructive cases correspond, respectively, to the limiting values $\Gamma \to 2$ as $\tau \to 0$ and $\Gamma \to 1$ as $\tau \to \infty$. Thus, in the regenerative case, maximization of $\eta(\mu)$ implies [26]

$$\{1 - (\lambda/r_v)^\delta\}\Gamma + \{(\Gamma - 1)(\lambda/r_v)^\delta + 1 - \delta\}\delta \ln(\lambda/r_v) = 0, \tag{18}$$

where $\mu_{\rm opt} = 2 - \delta$, with $0 \leqslant \delta < 1$. We then note the following regimes.

(i) For $\Gamma > \Gamma_{\min}$, where

$$\Gamma_{\min} = \frac{\ln(\lambda/r_v)}{\ln(\lambda/r_v) - 1 + (\lambda/r_v)^{-1}} \qquad (19)$$

and fixed λ/r_v, the solution of equation (18) leads to a single peak in η at $1 < \mu_{\mathrm{opt}}(\tau) \leqslant 2$, so that μ_{opt} decreases with respect to the non-destructive case.

(ii) For $\Gamma < \Gamma_{\min}$, η presents a maximum at $\mu = 1^+$, decreasing monotonically for $\mu > 1$, similarly to the destructive case, where $\Gamma_{\min}$ relates to the revisitability threshold for there to be a peak in $\eta(\mu)$ for a physical value of μ.

The above results have been confirmed by numerical simulations [23, 25, 26] in one and two dimensions incorporating rules (1)–(2), with the advantage that no approximations are made. Interestingly, both theoretical arguments and numerical results suggest that the above conclusions might hold in any dimension of the search space (see, e.g., figure 2). Moreover, equations (8) and (11) describe the correct scaling properties even in the presence of short-range correlations in the directions and lengths of the flights. Hence, learning, predator–prey relationships and other short-term memory effects present in realistic biological foraging may become unimportant in the long-time long-distance limit [27].

We can further ask how the above results are affected if mobility of target sites and relative size of the searcher and targets are incorporated into the random search model. Numerical results [29] for destructive searches in one dimension, with periodic boundary conditions, indicate that, for low target densities, Lévy search processes are indeed better than Brownian for a wide range of velocity and size ratios. Actually, for Brownian target motion, a Lévy searcher is, in general, more efficient than a Brownian searcher, as similarly found when targets are static. In this 1D context, only for the extreme cases of targets almost one order of magnitude faster and larger than the searcher is the Brownian search as efficient as a Lévy search. Conversely, when the targets perform Lévy random walks, there is a wide range of velocity and size ratios for which Brownian searches are at least as efficient as Lévy searches. Therefore, Lévy target motion generally diminishes the efficiency of Lévy searches in comparison to Brownian targets, except when the searcher moves much faster than the target. (In the case of moving targets, a particularly efficient strategy for the searcher is to remain static and just wait to be 'found' by the target. In the context of animal foraging, this corresponds to ambushing and parasitelike behavior. However, such a strategy seems to be restricted to a limited number of species [3, 29, 30].) In addition, for Brownian targets, velocity and size ratios are equally relevant to the search efficiency, whereas for Lévy targets, size in 1D becomes more important than velocity [29]. Finally, in the regime of high target densities, Brownian and Lévy strategies of the searcher appear as optimally equivalent.

3. Lévy flights and walks on an interval with absorbing boundaries

Consider a Lévy flight that starts at point x_0 in a one-dimensional search space with boundaries (absorbing target sites) at $x = 0$ and $x = \lambda$ [20]. In the preceding section, the efficiency of the search was written in terms of the inverse of the average distance traversed between two successive target sites, $\langle L \rangle$, equation (4). In the present context, the cases of non-destructive and destructive searches correspond, respectively, to $x_0 = \ell_0$ (once having found the site at the origin, the searcher restarts at the minimum possible distance from it, if we set $r_v = \ell_0$) and $x_0 = \lambda/2$ (since the last visited site is destroyed, the searcher restarts on average at the middle of the interval). We now proceed to the calculation [20] of $\langle L \rangle$ and the mean number of flights between two successive target sites, $N \equiv \langle n \rangle$.

The probability density $P_n(x)$ of finding the Lévy searcher at point x after n flights can be expressed recursively in terms of initial probability density, $P_0(x) = \delta(x - x_0)$, as [20]:

$$P_n(x) = \left[\mathcal{L}_\mu^n P_0\right](x). \tag{20}$$

Here the integral operator [20] $\mathcal{L}_\mu$, with a kernel $p(x - y)$ given by equation (3), acts on a function $f(x)$ as

$$[\mathcal{L}_\mu f](y) = \frac{(\mu - 1)\ell_0^{\mu-1}}{2} \int_0^\lambda \frac{f(x)\theta(|x - y| - \ell_0)\,\mathrm{d}x}{|x - y|^\mu}. \tag{21}$$

A 'negative' step length, $-|x - y|$, means a step taken to the left, and $\theta(x) = 1$ for $x > 0$ or 0 otherwise. By considering the probability that the flyer is absorbed (a target is found, either at $x = 0$ or $x = \lambda$) exactly on flight n:

$$\tilde{P}_n = \int_0^\lambda \left[\left(\mathcal{L}_\mu^{n-1} - \mathcal{L}_\mu^n\right)P_0\right](x)\,\mathrm{d}x, \tag{22}$$

and integrating along with the Dirac delta function of P_0, we find a closed-form expression for the average number of flights before absorption [20]:

$$N \equiv \langle n \rangle = \sum_{n=1}^\infty \tilde{P}_n n = [(\mathcal{L}_\mu - \mathcal{I})^{-1}h](x_0), \tag{23}$$

where $\mathcal{I}$ is the unity operator and $h(x) = -1$. Equation (23) can generally be rewritten in terms of a Fredholm integral equation of the second kind with the kernel $p(x_0 - x_1)$ [31]:

$$Q(x_0) = \langle q_0(x_0) \rangle + \int_0^\lambda Q(x_1)p(x_0 - x_1)\,\mathrm{d}x_1, \tag{24}$$

where

$$Q(x_0) = \left\langle \sum_{i=1}^\infty q_i \right\rangle, \qquad q_i = q(x_{i-1}, x_i) \tag{25}$$

is a function of the starting point x_{i-1} and the ending point x_i of flight i; $\langle\ \rangle$ denotes the average over all possible processes starting at x_0, and

$$\langle q_0(x_0) \rangle \equiv \int_{-\infty}^\infty p(x_1 - x_0)q(x_0, x_1)\,\mathrm{d}x_1. \tag{26}$$

If x_1 lies outside the interval $[0, \lambda]$, the searcher is absorbed by one of the boundaries and the value of $q(x_0, x_1)$ should be defined according to its physical meaning in the search process. In particular, the above calculation corresponds to the assignments $Q(x) = \langle n(x) \rangle$ and $\langle q_0(x) \rangle = -h(x)$. Moreover, a similar procedure using $Q(x) = \langle L(x) \rangle$ and $\langle q_0(x) \rangle = \langle |\ell(x)| \rangle$ allows one to write the average distance traversed between two successive target sites [20]:

$$\langle L \rangle = -[(\mathcal{L}_\mu - \mathcal{I})^{-1}\langle |\ell| \rangle](x_0). \tag{27}$$

Here the mean length of a single step that starts from a point y in an interval $[\ell_0, \lambda - \ell_0]$ reads

$$\langle |\ell(y)| \rangle = \frac{(\mu - 1)\ell_0^{\mu-1}}{2}\left[\int_0^{y-\ell_0} \frac{\mathrm{d}x}{(y - x)^{\mu-1}} + \int_{y+\ell_0}^\lambda \frac{\mathrm{d}x}{(x - y)^{\mu-1}} \right.$$
$$\left. + y\int_{-\infty}^0 \frac{\mathrm{d}x}{(y - x)^\mu} + (\lambda - y)\int_\lambda^\infty \frac{\mathrm{d}x}{(x - y)^{\mu-1}}\right]. \tag{28}$$

This equation suitably incorporates the possibility of truncation of the flight by the encounter of one of the targets at the boundaries. In order to perform the numerical integration of equation (24) in the cases $Q(x) = \langle n(x) \rangle$ and $Q(x) = \langle L(x) \rangle$, we replace [20] the integration
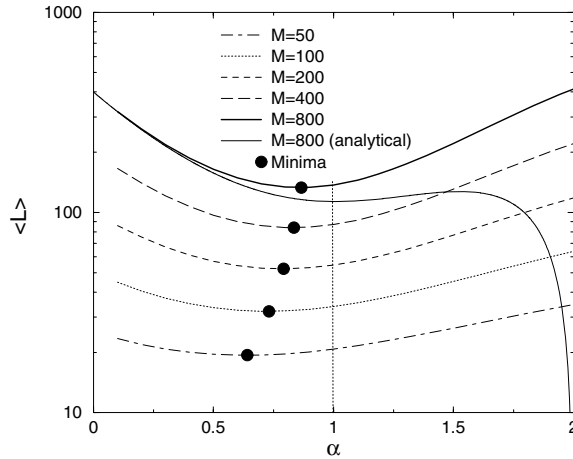
Figure 3. Semi-log plot of the numerical solutions of the average distance $\langle L \rangle$ traversed by the searcher before finding a target site at one of the boundaries in one dimension, as a function of $\alpha = \mu - 1$, for $x_0 = \ell_0 = 1$ (non-destructive search) and various values of $M \equiv \lambda/\ell_0$ [20]. Circles indicate the positions of the minima of $\langle L \rangle$ (maximum efficiency η), which shift toward $\alpha = 1$ ($\mu = 2$), as M increases. The analytical continuous limit approximation, equation (37), is also shown for $M = 800$.

by a summation and the kernel $p(x - y)$ by the matrix A_{ij}, with $0 < i < M \equiv \lambda/\ell_0$, such that $A_{ii} = 0$ and

$$A_{ij} = \frac{1}{2} \left[\frac{1}{|i-j|^{\mu-1}} - \frac{1}{(|i-j|+1)^{\mu-1}} \right], \qquad i \neq j. \tag{29}$$

The results for $\langle L \rangle$ in the non-destructive case ($x_0 = \ell_0 = 1$) are shown in figure 3 [20]. Note that, as $M = \lambda/\ell_0$ gets larger (i.e. for low density of targets), the minimum in $\langle L \rangle$ (maximum in the efficiency $\eta = \langle L \rangle^{-1}$) approaches the value $\alpha = 1$ ($\mu = 2$), in agreement with the results of the preceding section; in contrast, for $x_0 = \lambda/2$ (the destructive case) it has been found [20] that the minimum in $\langle L \rangle$ tends to $\alpha = 0$ ($\mu = 1$), as expected.

It is also interesting to consider the continuous limit [20] of the random search in which, instead of a sequence of discrete steps, the probability density of finding the Lévy searcher at point x evolves continuously with time according to the superdiffusion equation [32]

$$\frac{\partial P(x,t)}{\partial t} = \frac{\ell_0^{\mu-1}}{t_0} \mathcal{D}_\mu P(x,t), \tag{30}$$

where t_0 is the constant duration of each flight, $\ell_0^{\mu-1}/t_0$ is the fractional analog of the diffusion coefficient, and

$$\mathcal{D}_\mu \equiv \lim_{\ell_0 \to 0} \ell_0^{-\mu+1} [\mathcal{L}_\mu(\ell_0) - \mathcal{I}]. \tag{31}$$

Formal substitution of equation (31) into equation (23) leads to the following closed-form expression for the average time spent by the continuous Lévy flight process before absorption [20]:

$$\langle t \rangle = t_0 \langle n \rangle = \frac{t_0}{\ell_0^{\mu-1}} \left[\mathcal{D}_\mu^{-1} h \right](x_0) = \frac{t_0}{\ell_0^{\mu-1}} g(x_0). \tag{32}$$

Here $g(x)$ satisfies

$$\mathcal{D}_\mu g(x) = h(x) = -1, \tag{33}$$

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with boundary conditions $g(0) = g(\lambda) = 0$. Equation (33) belongs to a class of generalized Abel integral equations with the Riesz fractional kernel [33], whose solution for $1 < \mu \leqslant 3$ implies

$$g(x) = \frac{\sin[\pi(\mu - 1)/2]}{\pi(\mu - 1)/2}[(\lambda - x)x]^{(\mu-1)/2}. \tag{34}$$

For $\mu > 3$, equation (30) should be replaced by the standard diffusion equation, with the diffusion coefficient $D = (\mu - 1)\ell_0^2 / [2t_0(\mu - 3)]$, so that

$$\langle t \rangle = x(\lambda - x)/(2D). \tag{35}$$

In the limit $\ell_0 \to 0$, the formal expansion of $(\mathcal{L}_\mu - \mathcal{I})^{-1}$ allows one to identify [20] $\ell_0^{-\mu+1} g(x_0)$ as the first term of the expansion of the average number of flights $\langle n \rangle$ of the discrete process. Since the whole formal expansion may not converge, its first term is then defined [20] as the average number of flights for the continuous process:

$$n_\mu(x_0) \equiv \ell_0^{-\mu+1} g(x_0) = \frac{\sin[\pi(\mu - 1)/2]}{\pi(\mu - 1)/2} \left[\frac{(\lambda - x_0)x_0}{\ell_0} \right]^{(\mu-1)/2}. \tag{36}$$

This result agrees with equations (8) and (11), respectively, for $x_0 = \lambda/2$ (destructive search) and $x_0 = \ell_0 = r_v \ll \lambda$ (non-destructive search), which have been derived in section 2 using scaling arguments. In particular, for $x_0 = \lambda/2$ has been found [20] good convergence of the scaled average number of flights of the discrete process, $\langle n \rangle (M/2)^{-\mu+1}$, with $\langle n \rangle$ as in equation (23), to the analytical continuous limit expression, equation (36).

Finally, in analogy to the procedure above, an analytical expression for the continuous limit approximation of the average total path length exists [20] in terms of hypergeometric functions F:

$$L_\mu(x_0) = \frac{\lambda(3 - \mu)}{2(2 - \mu)} \left[1 - 4 \frac{\psi_\mu(z) + \psi_\mu(1 - z)}{(\mu - 1)(\mu + 1)B((\mu - 1)/2, (\mu - 1)/2)} \right]$$
$$+ \frac{2\lambda M^{\mu-2} \sin(\pi(\mu - 1)/2)(z - z^2)^{(\mu-1)/2}}{\pi(\mu - 2)}, \tag{37}$$

where $z = x_0/\lambda$, $B(a, b) = \Gamma(a)\Gamma(b)/\Gamma(a + b)$ is the Euler B function,

$$\psi_\mu(z) = F\left(2 - \frac{(\mu - 1)}{2}, \frac{(\mu - 1)}{2}, \frac{(\mu - 1)}{2} + 2, z \right) z^{(\mu-1)/2+1}, \tag{38}$$

and

$$F(a, b, c, x) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(n + a)\Gamma(n + b)x^n}{\Gamma(n + 1)\Gamma(n + c)}. \tag{39}$$

Interestingly, in the non-destructive continuous case, L_μ presents two minima, at $\mu \to 2$ and $\mu = 3$ (see figure 3). The former is consistent with the maximum in the efficiency η discussed in section 2, whereas the latter does not exist in the discrete search process. We should mention, however, that the approximations made in the limit $\ell_0 \to 0$ break down [20] when $\mu \to 3$, in the vicinity of the absorbing boundaries, at $x_0 = 0$ or $x_0 = \lambda$.

4. Lévy random searches in regular and defective lattices

The interest in the random search problem is not limited to the context of continuous (Euclidian) landscapes. This subject is also of relevance to networks and discrete search spaces in general [34, 35]. In many applications [36], such as the Internet, memory search in neural networks [37] and gene networks [38], the search environment is discrete or digital rather than continuous.

A number of studies [39–41] have confirmed the widespread hypothesis that the features of a lattice random search are related to the topological characteristics of the lattice, particularly the site connectivity. In this respect, one can partition lattice networks into two classes: (i) small-world and (ii) large-world networks. Small-world networks are globally connected and possess power-law link distance distributions. The lattice diameter grows sublinearly (logarithmically) [40, 42] with the number of sites due to the existence of rare long-range links—the ultra-long links act as shortcuts that reduce the number of links necessary across which the information must propagate (an example being the World Wide Web [43]). In contrast, large-world networks are locally connected and rich in clusters, with only short-range links, typically between first neighbor sites. In such cases, the distribution of link sizes is a narrow Gaussian, truncated close to the origin, giving rise to a linear increase of the mean distance between two arbitrary sites (lattice diameter) with the total number of sites [40, 42]. As a consequence, large-world lattices, usually associated with more inefficient systems for transport [40], are much more sensitive than small-world networks to the exact method used for looking for randomly distributed target sites, in analogy with the continuous case. Further, large-world networks and Euclidian landscapes also share the property that the farther a target site is, the larger is the traveled distance to reach it. This is not true, e.g., for non-scaled small-world lattices, in which a single bond may connect two extremely distant nodes.

We consider in the following the problem of random searches in regular and defective lattices [34, 35], with only local connections between nodes. As relevant factors determining the final outcomes of a random search in such environments, we analyze the role of connectivity, boundary conditions, density of targets and density of defects. Defects can be introduced in these search spaces by diluting the network, i.e., by randomly eliminating a certain fraction of nodes from an initially regular lattice (we have considered triangular and square lattices in [35]). Hence, if n_0 is the initial number of nodes in a perfect lattice, then a fragmentation coefficient can be defined by

$$\chi = n_d/n_0, \tag{40}$$

where the number of defects n_d is equivalent to the total number of nodes removed ($\chi = 1$ implies an empty space—total dilution, complete destruction of the lattice).

There are many possible choices for the boundary conditions. Three of particular interest are [35]: periodic (PBC), helical (HBC) and (not-specular) wall (WBC). In figure 4, they are depicted in a defect-free triangular geometry, and it is also illustrated how a searcher always moving straight forward in a single very long step can 'scan' the lattice. For the PBC we notice that horizontal paths are closed circles, whereas diagonal paths form solenoid-like curves. Indeed, the trajectory $ABCDE\ldots$ displayed in figure 4(*a*) is not closed until all the nodes are visited. The HBC differs from the PBC because the left (bottom) border is connected to the corresponding right (top) border shifted by one node. Therefore, the horizontal paths create a solenoid that is closed after half of the nodes have been visited. The diagonal paths also create a solenoid, which, analogously to the PBC case, closes only after all nodes are visited. Finally, in the WBC any path hitting a border node becomes truncated and a new path, with the direction randomly chosen, begins.

We now apply to the random searches in discrete spaces [34, 35] model rules equivalent to those described in section 2 for continuous landscapes. In the presence of defects, however, an extra rule is necessary: if a defective site exists along the direction of step j (within the distance ℓ_j), and no target site is found before reaching it, then the step is truncated at the node immediately before such a defect. In the sequence, a new direction and step length are chosen and the process resumes. Note that the encounters of a target site or a defective site can formally be treated alike in the efficiency function. The important
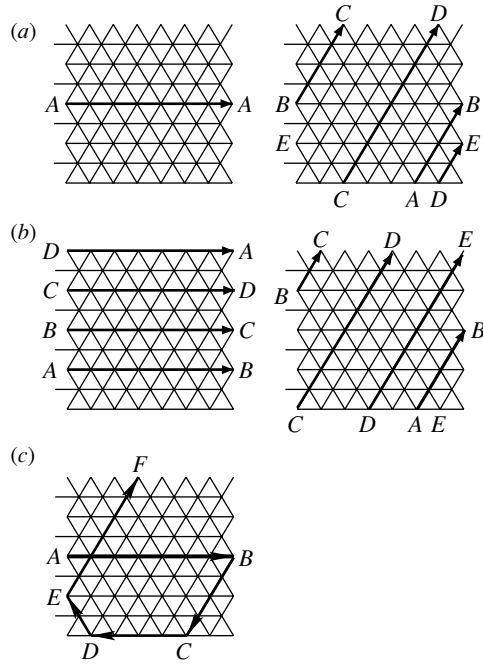
Figure 4. Three boundary conditions in the case of a defect-free triangular lattice [35]: (*a*) periodic (PBC), (*b*) helical (HBC), and (*c*) not specular wall (WBC).

difference is that the encounter of a defect does not add any gain to the searcher. In this sense, equations (4) and (5) should be replaced by [35]

$$\eta = \frac{p_t}{\langle L \rangle} \approx \frac{p_t}{N \langle \ell \rangle}, \tag{41}$$

where p_t denotes the probability of encounter of a target site ($p_t = 1$ for a defect-free lattice), and in this context $\langle L \rangle$ and N represent the respective mean distance traversed and number of steps performed between two successive encounters. In addition, we observe that

$$1/\lambda = 1/\lambda_t + 1/\lambda_d, \tag{42}$$

where λ_t and λ_d represent the mean free path ($\propto$ inverse density) of the lattices of targets and defects, respectively, defined as a function of the inverse number of sites n_t and defects n_d.

In the absence of defects, numerical results [34] using the PBC, in three regimes of target densities (low, intermediate and high), and destructive and non-destructive searches, have pointed out that local connectivity generally affects the search efficiency more drastically only at low densities. Actually, in this regime the triangular lattice provides higher outcomes than the square one, regardless the values of μ. Although most of the results for defect-free lattices look like those of Euclidian landscapes, we note that the possibility of loop-like trajectories for a single step with the PBC (see figure 4) causes the search efficiency to decrease for $\mu < 2$ in low density of targets, even in the destructive case, a feature not observed in searches performed in continuous environments (section 2). Therefore, in this low-density regime with the PBC, destructive and non-destructive search efficiencies in defect-free lattices become alike. Of course, this result is not expected to occur for boundary conditions in which the existence of such long loops is naturally prevented.
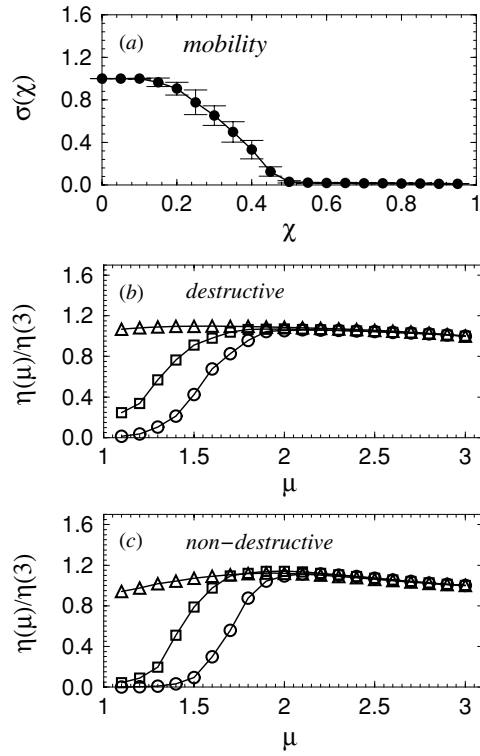
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Figure 5. (*a*) Fraction of quadrants visited at least once by the searcher as a function of the fragmentation coefficient, averaged over 100 search runs [35]. (*b*)–(*c*) Normalized efficiency as a function of μ for (*b*) destructive and (*c*) non-destructive searches, in the regime of low density of defects ($r_v = 1$, $n_0 = 4 \times 10^6$, $n_t = 10^3$, $n_d = 10^3$, $\lambda/\ell_0 = 1331.33$ and total distance traveled $L = 10^5 \lambda$), for the PBC (circles), HBC (squares) and WBC (triangles).

When defects are introduced, a first interesting result arises, as shown in figure 5(*a*) [35]. By dividing the lattice into identical quadrants, the fraction of quadrants visited during the search, related to the mobility of the searcher while scanning the space, becomes null (the searcher gets locally trapped) above some typical value of the fragmentation coefficient ($\chi \approx 0.5$ for triangular lattices). Such a value coincides with the critical concentration of the lattice percolation problem [44], indicating that the fraction of quadrants visited by the searcher is directly connected to how close the defective lattice is to its percolation limit. Indeed, searches in lattices with high density of defects are extremely inefficient. Therefore, in the following we restrict ourselves to the analysis in the regime of low densities of defects. What the exact threshold is for a small density of defects is a very relevant, but still open, question.

In low densities of defects, the shape of the efficiency curve is strongly dependent on the density of targets and the boundary condition used [35]. For low target densities, the lattice limits are often reached, and the boundary condition used becomes relevant. In this regime, a situation similar to the loop trajectories discussed above occurs for the PBC and HBC, but not for the non-specular case of WBC. Thus, instead of decreasing for $\mu < 2$ even in the destructive case, as found for the PBC and HBC, it can be seen in figure 5(*b*) that the efficiency using the WBC reaches a maximum close to $\mu = 1$, similarly to the continuous

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searches. In the intermediate target density regime, when the truncation rate is already high, the lattice borders are rarely accessed in a single step, and η becomes essentially independent of the boundary condition used. At last, for high density of targets, practically every step is truncated and any search strategy is as much efficient as the Brownian one.

5. A necessary criterion for distinguishing true superdiffusion from correlated random walk processes

Lévy processes and CRW are among the most studied random walk models proposed as the best strategy in the random search problem [1–8, 17, 18]. Although very distinct from the theoretical point of view, severe difficulties arise when a clear-cut conclusion about which strategy is being used should be drawn from a finite set of experimental data. Regarding the dynamics of the search process, due to their Markovian nature CRW tend asymptotically to Brownian (normal) diffusion for times beyond some typical characteristic correlation time, $t > \tau$. In spite of this, on time scales $t < \tau$ apparent superdiffusion takes place in CRW, due to, e.g., retention of directional memory. On the other hand, uncorrelated scale-invariant Lévy walks present genuine superdiffusive (anomalous) dynamics for $1 < \mu \leqslant 3$ on all time scales, and ballistic motion in the limit $\mu \to 1$. Even truncated Lévy processes, such as those described in the preceding sections, retain the most important properties of non-truncated Lévy walks and flights to a considerable extent [24]. In most of the applications, including animal foraging, experimental data sets typically spanning only a few decades of time (or length) scales may point to both Lévy processes and CRW as adequate random walk models to describe the random search properties [45]. As a consequence, an important issue regards the development of methods and techniques to clearly detect the evidence of one of these two search strategies from experimental data. For example, there are many methods applied to probe the existence of correlations and superdiffusion in finite time series, such as those using moving data windows along with detrended fluctuation analysis and wavelets [46]. In the former case, for instance, difficulties arise related to the arbitrariness of the procedure that transforms a recorded continuous search path into a discretized random walk [47]. Another example involves a technique to correct misleading conclusions regarding the presence of multifractal anomalous diffusion, potentially guessed from finite data set windows [48]. These situations present in common the necessity to determine what is the minimum amount of data needed to reach credible conclusions on the strategy underlying a given random search process. In the following, we discuss a necessary but not sufficient condition for distinguishing true superdiffusion from correlated random walk processes [49].

Consider two-dimensional CRW models in which persistence is controlled by the probability distribution of relative turning angles (i.e. directional memory). To obtain the correlation time (or length) scale, we define [49] an adimensional two-point correlation function as $C(|j - i|) \equiv \langle \mathbf{r}_j \cdot \mathbf{r}_i \rangle / \langle r_j r_i \rangle$, where $\mathbf{r}_j$ is the vector position of the searcher after step j. If the distributions of step lengths r_j (of finite variance) and turning angles θ_j are independent, thus the correlation between successive steps reads

$$C(1) = \langle \cos(\theta) \rangle = \int_{-\pi}^{\pi} \mathrm{d}\theta \, \cos(\theta) \, f_w(\theta), \tag{43}$$

where f_w denotes the circular 'wrapped' probability density function of relative turning angles. For circular statistics [50], the mean resultant length ρ and mean direction $\bar{\theta}$ are related to the first circular characteristic function by $\phi_1 \equiv \langle \exp(\mathrm{i}\theta) \rangle = \rho \exp(\mathrm{i}\bar{\theta})$, so that, in the symmetric $\bar{\theta} = 0$ case, one has $\rho = C(1)$ (note that $\rho = 0$ and $\rho = 1$ for the extreme cases of uniform

and Dirac $\delta(\theta)$ distributions of turning angles, respectively). The Markov character of a CRW implies

$$C(t/t_0) \sim [C(1)]^{t/t_0} = \exp\{(t/t_0)\ln[\langle\cos(\theta)\rangle]\}, \tag{44}$$

where t_0 is the typical time of one step, and the adimensional correlation time (or length) is identified as $\tau = -1/\ln[\langle\cos(\theta)\rangle]$. This means that for any time scale a few orders of magnitude larger than τ, the CRW appears Brownian (normal diffusion), since the searcher cannot keep successive step orientations correlated. Therefore, any experimental data set spanning a time period Δ not larger then the correlation time does not contain sufficient information to probe superdiffusive behavior with any level of statistical significance. A necessary condition for establishing superdiffusion thus can read [49]

$$\Delta \gg \tau_{\text{meas.}} = -1/\ln[\langle\cos(\theta)\rangle_{\text{meas.}}], \tag{45}$$

where $\langle\cos(\theta)\rangle_{\text{meas.}}$ denotes the experimental average measure. Since CRW can also have additional correlations besides directional persistence, this criterion cannot be a sufficient condition, and the above calculated $\tau_{\text{meas.}}$ might actually represent only a minimum lower bound for the correlation time in more general circumstances. Thus, even if the data set satisfies equation (45) and appears superdiffusive on scales much larger than $\tau_{\text{meas.}}$, yet this is not sufficient to rule out diffusive behavior at even larger scales, and complementary tests of superdiffusion should be applied to eliminate possible spurious false positives.

In [49], the above condition is successfully tested using two random walk models with the same ρ and $\tau_{\text{meas.}}$, and identical wrapped Cauchy (WC) turning angle distributions, $f_{\text{WC}}(\theta)$, with $\rho = 0.9$, where

$$f_{\text{WC}}(\theta) = (2\pi)^{-1}(1 - \rho^2)/[1 + \rho^2 - 2\rho\cos(\theta)] \tag{46}$$

(see below): one with global Lévy-like properties, displaying genuine superdiffusive behavior at all time scales (a kind of 'curved' Lévy walk) and the other which behaves superdiffusively only at scales $t < \tau_{\text{meas.}}$.

The influence of the shape (relative kurtosis) and correlation length of the turning angle distributions on the long-term search efficiency of CRW has also been investigated in [51], for randomly uniform and patchy target site distributions. A number of CRW models has been considered, with angular probability density belonging to the family of circular symmetric unimodal distributions proposed by Jones and Pewsey [52] ($\kappa \geqslant 0$ and γ real):

$$f_{\text{JP}}(\theta; \gamma, \kappa) = \frac{[\cosh(\kappa\gamma) + \sinh(\kappa\gamma)\cos(\theta)]^{1/\gamma}}{2\pi\, P_{1/\gamma}^0(\cosh(\kappa\gamma))}, \tag{47}$$

where $P_\beta^n(.)$ denotes the associated Legendre function of first kind of order n and degree β. Equation (47) reduces to some well-known cases, such as the von Mises [$\rho = \langle\cos(\theta)\rangle = I_1(\kappa)/I_0(\kappa)$], cardioid [$\rho = \tanh(\kappa)/2$] and WC distributions [$\rho = \tanh(\kappa/2)$], by setting, respectively, $\gamma = 0$, $\gamma = -1$ and $\gamma = +1$. For other values of γ,

$$\rho = |\gamma| P_{1/\gamma}^1(\cosh(\kappa\gamma))/[(1 + \gamma) P_{1/\gamma}^0(\cosh(\kappa\gamma))]. \tag{48}$$

It has been found [51] that, in environments with uniformly random target site distributions, the correlation length has a large influence in the search efficiency in any regime of target density. In this case, similar η can be found even for distinctly shaped angular distributions, provided that ρ is the same. In contrast, in patchy landscapes the particular shape of the distribution also becomes relevant, specially at high target densities. Excessively sharp distributions are found to generate very inefficient searches when local target density fluctuations are large.

6. Energy considerations and critical survival states on the edge of extinction

When the balance of energy income (due to the finding of targets) and expenditure along the search walk is incorporated into the energy efficiency of the search, η_E, defined as the ratio between the mean net energy gained by the searcher and the average total length of the search walk, it can be shown [25, 26] that, for any arbitrary energy cost function, η_E and the statistical efficiency η, discussed in section 2, present maximum values at the same μ_{opt}. However, the constraint of positive energy along the walk may lead such an optimal strategy to be restricted in realistic situations. Actually, the range of allowed values of μ such that $\eta_E > 0$ is found to be dependent on specific parameters of the energy cost function, as well as on the density of target sites. For a given choice of an energy cost function, it has been found [26] that there exists some typical value of λ above which the random searcher enters an absorbing (extinction) state of non-positive net energy, from which it cannot escape.

These issues have been addressed in more detail in [53], in which the net energy accumulated by the searcher along the search path is studied in the context of a destructive search for a moving target in one dimension. Actually, such a problem belongs to the general family of reaction–diffusion processes [54, 55], to which local density fluctuations and the nature of the diffusion mechanism become important in low dimensions, in contrast to the fluctuation-insensitive mean-field-like behavior observed at high dimensions [54, 56–58].

In [53], the rules described in section 2 have been applied to both the searcher and moving target, which can actually take their step lengths from equation (3) with independent values of μ. In addition to those rules, the energy loss of the searcher along the path is assumed to be proportional to the step length (more complicated nonlinear cost functions do not seem to affect the qualitative behavior of the energetic search efficiency [26]). The energy obtained by the searcher after N flights is [53]

$$\xi_N = \xi_0 + \sum_{i=1}^{N} (g\delta_i - \alpha_i), \tag{49}$$

where ξ_0 is the initial energy, $\alpha_i = \alpha\ell_i$ in the linear cost energy function, g is the energy gain by the finding of a target, and $\delta_i = 1$ ($\delta_i = 0$) whenever there is (there is not) an encounter at flight i. By allowing the searcher to look for targets in a path with fixed total length L, we have observed [53] in the regime of low density of targets that generally $\xi_N < 0$ even before the path is completed. In this case, the searcher enters an absorbing state and the search walk is terminated. In fact, as shown in figure 6(a), for each pair of Lévy indices chosen for the searcher and target, there is a well-defined value of g/λ below which no search walk (in an average over 500 runs) survives with $\xi_N > 0$ after traversing a total distance $L = 10^4$. If such a value of g/λ (the threshold of extinction) marks a critical point of a phase transition involving absorbing states [54] is a question under present investigation.

From equation (49) and considering static targets, it is found [53] that

$$\frac{\xi_N - \xi_0}{L} \approx \mathcal{N}\{g[\langle\ell\rangle_t + (N_d - 1)\langle\ell\rangle_{nt}]^{-1} - \alpha\}, \tag{50}$$

where $\langle\ell\rangle_t$ ($\langle\ell\rangle_{nt}$) is the average length of a single flight in the case it is truncated (not truncated), and $\mathcal{N} \sim (\lambda/r_v)^{\mu-3}$ is a multiplicative factor to account for the possibility of aborting the run when the net energy reaches zero; its exponent dependence, $\mu - 3$, is understood since the occurrence of statistical fluctuations is irrelevant at the Gaussian $\mu > 3$ regime of relatively shorter flight lengths. Despite the limitation of considering static targets, the analytical model results, displayed in figure 6(b), agree qualitatively [53] with the numerical simulation, figure 6(a), except for the case of nearly ballistic dynamics of the searcher ($\mu = 1.1$).
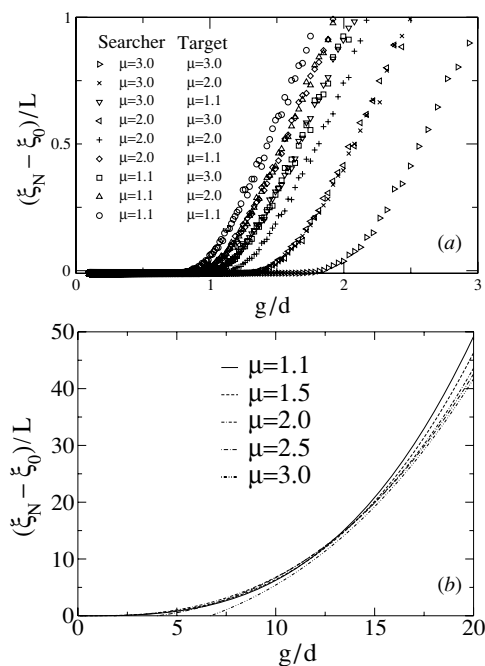
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Figure 6. The energetic gain as a function of the renormalized effective density g/d ($d \equiv 2\lambda$), for various values of μ of the searcher and moving target, and parameters $\xi_0 = 100$, $g = 100$, $\alpha = 1$, $r_v = 1$, $L = 10^4$ and unity velocities, as found [53]: (*a*) from numerical simulations (averaged over 500 search runs) and (*b*) from the approximate analytical model considering a static target (see the text).

Generally speaking, superdiffusion of either the searcher or target confers substantial energetic advantages to the former [53]. Most importantly, while superdiffusion does not appear to play a relevant role for high target densities, in contrast, it leads to a crucial advantage in the limit of low densities—i.e. at the edge of extinction. Actually, Brownian searches rapidly enter the absorbing state, but superdiffusive searches remain with positive net energy for the long term. It is interesting to remark that, in the context of animal foraging, recent empirical data [59] indicate that some species actually increase their diffusiveness in regimes near starvation, when compared to their foraging activity under normal conditions, in agreement with the above results.

7. Emergence of Lévy behavior in deterministic walks

So far we have discussed random search walks with inherently stochastic dynamics. This means that, beyond its presence in the target site distribution, we have also assumed randomness from *a priori* in the rules of motion. A less widely studied class of problems concerns deterministic walks [60–63], in which the rules of locomotion are taken from some purely deterministic model, rather than involving probability distributions of step lengths. The stochastic character of the search in this case is constrained to the randomness of the target distribution. In this context, it has been pointed out [64, 65] that, by interacting with complex environments, deterministic walks can effectively lead to the emergence of a variety of memory effects, angular (i.e. orientational) correlations and scale-free properties. In particular, scale-invariant Lévy-like distributions of effective step lengths can take place when the searcher looks
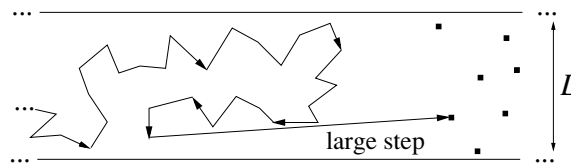
Figure 7. Deterministic search walks in rectangular environments with randomly distributed targets, where the 'go-to-the-nearest-site' rule leads to power-law distributions of move lengths for specific critical aspect ratios for L (strip-like geometry). This surprising result is due to the fact that in the critical condition the walker can 'accidentally' back track, and ends up having to make an ultra-long flight to return to the forward more abundant region of target sites [65].

for randomly placed target sites in a two-dimensional landscape of fixed area, $A = 1 = L_x \times L$, with $L \leqslant L_x$, by moving according to the following deterministic rules [65]:

(1) Once at a certain target site, the searcher goes straight to the closest available site.
(2) The searcher does not come back to any previously visited target—the search is destructive, although, in contrast to section 2, the total number of target sites decreases as they are found along the walk.

It has numerically been found [65] that, in the one-dimensional ($L \to 0$) and two-dimensional ($L \to L_x$) limits, the step length distribution presents a characteristic scale and converging second moment, thus being governed by the central limit theorem (Poisson in the 1D case and power-law decaying with $\mu > 3$ in 2D), as expected. Remarkably, however, Lévy-like length distributions with $2 < \mu < 3$ have been observed in the 1D-to-2D crossover regime (strip-like geometry), for $2 < L/\ell_n < 30$, where ℓ_n is twice the initial mean distance between nearest neighbor target sites (for instance, $\mu \approx 2.2$ for $L/\ell_n \approx 4.22$). This behavior results [65] from a non-trivial dynamical trapping mechanism in which, although the destruction of previously visited sites makes the searcher tend to move forward with higher probability in a narrow strip-like geometry, there is also a small fraction of large turning angles along the walk, allowing the searcher to back track to visit target sites left behind. In this case, after some steps the searcher ends up in a region depleted of targets; it is then eventually forced to make an ultra-long jump to access again a region rich in unvisited sites. An illustration of such a mechanism is given in figure 7. One concludes that effective Lévy-like behavior can emerge as result of short-range deterministic dynamics rules of movement and the interaction with the (Brownian-like) disordered media presenting a geometrical constraint [65].

8. Biological foraging

A great deal of motivation to the random search problem has arisen in the context of biological foraging [1–8, 12–18], which generally involves the dynamics of encounter rates between individuals of the same species, including humans (e.g. mating, territorial competition) or inter-specific interactions between a 'consumer' and a 'consumable' of distinct species (e.g. search for food in natural environments, predation). In this sense, efficient strategy choices become essential in determining the survival and fitness of a given species, since inefficient foraging could deplete the energy reserves of individuals and give rise to rates of encounter below a minimum acceptable threshold, possibly leading the species to enter an absorbing extinction state. From an evolutionary perspective, it is certainly pertinent to ask if efficient encounter rates are driven from selective pressures that systematically force organism adaption toward some optimal (or sub-optimal) solution.

Regarding the nature of the searching drive, the finiteness of the forager's 'radius of vision' and the lack of information about the location of the target objects beyond the close vicinity of the forager imply that in many instances the search is guided almost entirely by external cues, related to detective (olfaction, vision, etc) or cognitive (memory, learning, etc) skills of the searcher. Therefore, the foraging dynamics that defines the final outcomes of biological encounters becomes in essence a stochastic process (e.g. searches in landscapes with low density of food sites). In addition, the justification for truncated step length or time distributions also arises naturally in the context of biological encounters, since infinitely long flights are not allowed for foraging organisms.

Currently, biological foraging has been a very active field of research, with Lévy flights and walks [1–8], as well as intermittent walks [13–16] and CRW (including biased and composite Brownian walks) [17, 18], appearing among the most studied random walk models proposed as the best foraging strategy to look for food sites. From a historical viewpoint, the earlier foraging hypothesis assumed [3–5] forager's step length (or time) distributions with characteristic length (or time) scales. Thus, Gaussian, Rayleigh and other classical (Brownian) distributions with finite moments have been initially considered as optimal search strategies. However, empirical evidence of long-tailed power-law distributions of flight lengths or times [1, 66–68], as well as the scale-invariant fractal structure of episodes of foraging activity and rest [69], in species as diverse as swimming microorganisms [66] and fruit flies [69], gave rise to the suggestion that biological organisms may perform Lévy walks while looking for nutrients. Empirical data on the foraging flight times of an albatross species [70] have also provided support to the Lévy hypothesis, and a random search model has been applied [23] to investigate why searches with the Lévy index $\mu \approx 2$ appear as optimal when food is scarce and there is no constraint to access the target sites (see section 2). Subsequent studies [64, 71] on several distinct species (from dinoflagellates to mammals and fish) also seem to give support to Lévy flight foraging [6]. However, a recent revisit [72, 73] to the empirical data of albatrosses has questioned the earlier conclusions, suggesting a (Brownian-like) shifted gamma distribution of flight times. In spite of this, the most extensive foraging study [74] up to this moment, involving more than 1 200 000 search moves of 31 individuals from 7 distinct species of marine predators all over the planet, recently concluded that 5 out of the 7 species investigated do perform Lévy foraging, with an average index $\overline{\mu} = 2.12 \pm 0.31$. The use [74, 75] of rank–frequency plots and comparison of different models in this study actually take into account the criticisms [72] of considering analysis based on double log plots of histograms. Refined statistical methods of data analysis have also been applied to recently conclude the existence of Lévy flight searches by honeybees [76], carabid beetles [77] and drosophila fruit flies [78]. In addition, large-scale migration movements of five species of birds (including large ones such as storks) have also been found to be compatible with Lévy flights [79]. On the other hand, recent studies on the zooplankton daphnia [80] and ants [81] have claimed to provide support to single and composite Brownian walk models, respectively. As one can see, the debate on the random search problem in the context of biological foraging is still open. However, the accumulation of recent empirical and theoretical evidence seems to have shifted the question from 'do animals perform Lévy searches?' to the issue of 'under which circumstances and how often do animals perform Lévy searches?' [7, 82, 83].

9. Discussion and conclusions

Lévy processes stand, along with intermittent walks and correlated random walks, as the most studied random walk models proposed as the best strategy in the random search problem. In the past three decades, a great richness of concepts and methods of statistical physics has been

employed in the attempt to unveil the mechanisms underlying efficient searches. Among them we cite scaling ideas, fractal geometry, correlation functions, first-passage time processes and superdiffusive dynamics.

In this work, we have reviewed some of the contributions by our group to the study of the random search problem, most of them considering Lévy processes in several regimes of target site density, mobility and revisitability. The analysis performed through mean-field-like and closed-form average calculations, as well as by using numerical simulations, includes random searches occurring in continuous environments, and also in regular and defective lattices, with stochastic or deterministic rules of motion.

The practical relevance of the random search problem arises in contexts as diverse as automated computer searches of registers in high-capacity databases, oil recovery from mature reservoirs and even human motion. In particular, the field of biological foraging has greatly benefited from the recent progress in the random search problem.

The still unsolved challenges to the understanding of efficient random searches may reveal in the near future still more profound and unexpected features of their very rich underlying dynamics. Theoretical progress as well as advances in the acquirement of empirical data will pave the way toward this goal.

Acknowledgments

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